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Bell and Zeno

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Bell's inequalities and related inequalities of Wigner, Clauser–Horne–Shimony–Holt, Accardi–Fedullo, Gudder–Zanghi, Herbert–Peres, Khrennikov, others, are shown to be contained within a general operator trigonometry developed by this author starting in 1967. These inequalities are improved here to useful quantum spin correlation identities. Secondly, the Zeno problems from quantum measurement theory are traced from early work by this author starting in 1974, to the present. A Zeno Alternative that stresses domain-theoretic properties as essential to distinguishing reversible from irreversible quantum evolutions is presented.

KEY WORDS: Bell's inequalities; nonlocality; quantum Zeno dynamics; operator theory; operator trigonometry.

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1. INTRODUCTION

Here we will take both Bell theory and Zeno theory from their beginnings approximately 40 and 30 years ago, respectively, to the present. Of course the ensuing developments in quantum mechanics related to Bell's inequalities, and to the Zeno measurement issues, are now both large enterprises. Thus, the presentation here will be very selective, and only from the author's point of view. It turns out, discovered by the author only in 1996, that much of the Bell theory for quantum mechanics may be placed within a general operator trigonometry which this author developed independently, starting in 1967. As to Zeno quantum dynamics, this author was involved in the early work in 1974, and has recently returned to that theory.

Section 2 will trace the Bell theory, and how it is contained within our operator trigonometry. Section 3 will present what we will call here quantum trigonometric identities, for lack of a better term. These make exact Bell-type inequalities. Section 4 then traces the Zeno theory from our original work, with emphasis on certain critical domain-theoretic issues that were not resolved then and that continue to be overlooked. Section 5 presents our proposed Zeno Alternative. This

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leads to a reconsideration of the von Neumann projection axiom and Ludwig's effect operators within a larger picture which we call that of measurors/preparors.

2. BELL THEORY

Addressing fundamental issues raised by Einstein *et al.* (1935) concerning the foundations of quantum mechanics (Von Neumann, 1932), Bell (1964) presented his famous inequality

$$|P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| \leq \mathbf{1} + P(\mathbf{b}, \mathbf{c})$$

and exhibited quantum spin measurement configurations whose quantum expectation values could not satisfy his inequality. Bell's analysis assumed that two measuring apparatuses could be regarded as physically totally separated, and free from any effects from the other. Thus, his inequality could provide a test which could be failed by measurements performed on correlated quantum systems. It was therefore argued that local realistic hidden variable theories could not hold in quantum mechanics if future physical experiments would violate Bell's inequality. Later, Aspect *et al.* (1982) indeed demonstrated violation of Bell's inequality in their laboratory experiments. But the controversy about the Bell inequality and related inequalities to be mentioned later and their physical implications, continues to this day.

Bell's arguments in arriving at his inequality were classical probabilistic correlation arguments. However, it is known and easy to prove that this inequality holds for any real numbers a, b, c in the interval $[-1, 1]$: then $ab - bc + ac \leq 1$. Here is a proof. From $b^2 \leq 1$ and $c^2 \leq 1$ we have $b^2(1 - c^2) \leq 1 - c^2$ and hence $b^2 + c^2 \leq 1 + b^2c^2$. Adding $2bc$ to both sides and multiplying by $a^2 \leq 1$ we therefore have $a^2(b^2 + c^2 + 2bc) \leq b^2 + c^2 + 2bc \leq 1 + b^2c^2 + 2bc$, that is, $a^2(b + c)^2 \leq (1 + bc)^2$. Taking the positive square root yields $a(b + c) \leq |a||b + c| \leq 1 + bc$.

Wigner (1970) presented his own version, making more clear the issues of locality and so-called realism. Furthermore, Wigner was sure to use quantum mechanical probabilistic correlations. His version of Bell's theory then becomes the inequality

$$\frac{1}{2}\sin^2\frac{1}{2}\theta_{23} + \frac{1}{2}\sin^2\frac{1}{2}\theta_{12} \geq \frac{1}{2}\sin^2\frac{1}{2}\theta_{31}$$

where the θ_{ij} are angles between spin directions ω_i and ω_k. There are three other similar inequalities to take account of all possible configurations but we may speak here only of this one.

Another important Bell-type inequality is that of Clauser *et al.* (1969). This one is favored by experimentalists. As one experimentalist at the Angstrom Institute in Uppsala told this author in 2002, "all one has to do is set it up and then turn the dial." Let **a**, **b**, **c**, **d** be four arbitrary chosen unit vector directions in plane orthogonal to the two beams produced by the source. Let $v_i(\mathbf{a})$ and $v_i(\mathbf{d})$ be the "hidden" predetermined values ± 1 of the spin components along **a** and **d**, respectively, of particle 1 of the ith pair; similarly $w_i(\mathbf{b})$ and $w_i(\mathbf{c})$ for particle 2 values along directions **b** and **c**. Then the average correlation value for particle 1 spins measured along **a** and particle 2 spins measured along **b** is $E(\mathbf{a},\mathbf{b}) = \sum_{i=1}^{N} v_i(\mathbf{a})w_i(\mathbf{b})/N$. Taking into account in the same way the average correlation values $E(\mathbf{a},\mathbf{c})$, $E(\mathbf{d},\mathbf{b})$, $E(\mathbf{d},\mathbf{c})$ and adding up all pairs, one arrives at the CHSH inequality

$$|E(\mathbf{a},\mathbf{b}) + E(\mathbf{a},\mathbf{c}) + E(\mathbf{d},\mathbf{b}) - E(\mathbf{d},\mathbf{c})| \le 2.$$

In this author's operator trigonometry, see the books (Gustafson, 1997; Gustafson and Rao, 1997) for more background, we needed the following triangle inequality. Let x, y, z be any three nonzero vectors in a real or complex Hilbert space of any dimension. We take $||x|| = ||y|| = ||z|| = 1$ for convenience. From $\langle x, y\rangle = a_1 + ib_1$, $\langle y, z\rangle = a_2 + ib_2$, $\langle x, z\rangle = a_3 + ib_3$, define angles $\phi_{xy}, \phi_{yz}, \phi_{xz}$ in $[0, \pi]$ by $\cos\phi_{xy} = a_1$, $\cos\phi_{yz} = a_2$, $\cos\phi_{xz} = a_3$. Then, $\phi_{xz} \le \phi_{xy} + \phi_{yz}$. This rather natural inequality is less easy to prove than it looks, but it can be established in several ways. However, the best way, in the author's opinion, is to form the Gram matrix

$$G = \begin{bmatrix} \langle x, x\rangle & \langle x, y\rangle & \langle x, z\rangle \\ \langle y, x\rangle & \langle y, y\rangle & \langle y, z\rangle \\ \langle z, x\rangle & \langle z, y\rangle & \langle z, z\rangle \end{bmatrix}.$$

Then its determinant $|G|$ satisfies

$$|G| = 1 + 2a_1a_2a_3 - \left(a_1^2 + a_2^2 + a_3^2\right) \ge 0,$$

with strict positivity iff the vectors x, y, z are linearly independent. From $|G| \ge 0$ it is easy to prove the triangle inequality. But the $|G| \ge 0$ inequality is even more fundamental and more important than the triangle inequality.

The author noticed in 1996 that the Accardi and Fedullo (1982) inequality

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 1 \le 2\cos\alpha\cos\beta\cos\gamma$$

is contained in our Gram determinant inequality given above. The quantum spin model with angles α, β, γ has a representation in a 2-dimensional complex Hilbert space iff these angles satisfy the Accardi–Fedullo inequality. Immediately this author saw that moreover, much of the Bell theory was implicitly already contained within the author's operator trigonometry. Without giving all details here, "triangular-like" results of Gudder and Zanghi (1984), Herbert (1975), Peres (1993), Khrennikov (2000), and those of Bell, Clauser, Horne, Shimony, Holt, Wigner, Accardi, Fedullo, among others, can be placed within and clarified in the operator trigonometry.

Consider for example Wigner's inequality above, for the case in which he proved it, namely, all three directions being coplanar. Then, our Gram matrix is singular, the determinant $|G| = 0$, and we may write the Accardi–Gustafson inequality above as the equality

$$\left(1 - a_1^2\right) + \left(1 - a_2^2\right) - \left(1 - a_3^2\right) = 2a_3(a_3 - a_1 a_2).$$

This is a sharpening of Wigner's inequality.

For more information about Bell theory and all its ramifications, see e.g., the book Afriat and Selleri (1999) and other sources. For more details about the relations between the Bell theory and the operator trigonometry, see the papers Gustafson (1999, 2000, 2001, 2003, 2003a).

3. QUANTUM SPIN IDENTITIES

Here we would like to come up to date on the author's notion of inequality equalities (Gustafson, 2003), an unconventional terminology which we will replace here by: quantum spin correlation identities, or more generally, just quantum trigonometric identities. In fact this notion is more general, and a larger theory needs to be worked out, inasmuch as we are really talking about trigonometric identities that hold for vectors (and operators) for general Hilbert space, motivated however by the quantum mechanical setting discussed in this paper.

As a first example, consider Wigner's version of the Bell inequality discussed above. For the three coplanar directions, our corresponding inequality equality given above, becomes in Wigner's quantum spin setting terminology

$$\begin{aligned}&\sin^2\left(\tfrac{1}{2}\theta_{12}\right) + \sin^2\left(\tfrac{1}{2}\theta_{23}\right) - \sin^2\left(\tfrac{1}{2}\theta_{13}\right)\\ &= 2\cos\left(\tfrac{1}{2}\theta_{13}\right)\left[\cos\left(\tfrac{1}{2}\theta_{13}\right) - \cos\left(\tfrac{1}{2}\theta_{12}\right)\cos\left(\tfrac{1}{2}\theta_{23}\right)\right]\end{aligned}$$

Violation of the conventionally assumed quantum probability rule $|\langle u, v\rangle|^2 \equiv \cos^2\theta_{u,v}$ for unit vectors u and v representing prepared state u to be measured as state v, is equivalent according to Wigner to the right side of this identity being negative. This is his Bell "violation" test. However, from our point of view, there is no violation, there is just a quantum trigonometric identity, valid for certain formulations of measurement of certain spin systems.

As a second example, let us consider the important CHSH inequality given above. Wishing now to preserve equality

$$\begin{aligned}|\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} + \mathbf{d}\cdot\mathbf{b} - \mathbf{d}\cdot\mathbf{c}| &= |\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) + \mathbf{d}\cdot(\mathbf{b}-\mathbf{c})|\\ &= |\|b+c\|\cos\theta_{a,b+c} + \|b-c\|\cos\theta_{d,b-c}|\\ &= |(2+2\cos\theta_{bc})^{1/2}\cos\theta_{a,b+c}\\ &\quad + (2-2\cos\theta_{bc})^{1/2}\cos\theta_{d,b-c}|\end{aligned}$$

Squaring this expression and writing everything quantum trigonometrically,

$$
\begin{aligned}
|\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{d}\cdot\mathbf{b}-\mathbf{d}\cdot\mathbf{c}|^2 &= (2+2\cos\theta_{bc})\cos^2\theta_{a,b+c}\\
&\quad +(2-2\cos\theta_{bc})\cos^2\theta_{d,b-c)}\\
&\quad +2(4-4\cos^2\theta_{bc})^{1/2}\cos\theta_{a,b+c}\cos\theta_{d,b-c}\\
&= 4\cos^2(\theta_{bc}/2)\cos^2\theta_{a,b+c}\\
&\quad +4\sin^2(\theta_{bc}/2)\cos^2\theta_{d,b-c}\\
&\quad +4\sin\theta_{bc}\cos\theta_{a,b+c}\cos\theta_{d,b-c}.
\end{aligned}
$$

In the above we used two standard trigonometric half-angle formulas. Now substituting the double-angle formula $\sin\theta_{bc} = 2\sin(\theta_{bc}/2)\cos(\theta_{bc}/2)$ into the above we arrive at

$$|\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{d}\cdot\mathbf{b}-\mathbf{d}\cdot\mathbf{c}|^2 = 4[\cos\theta_{bc}/2)\cos\theta_{a,b+c}+\sin(\theta_{bc}/2)\cos\theta_{d,b-c}]^2$$

and hence the quantum *CHSH equality*

$$|\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{d}\cdot\mathbf{b}-\mathbf{d}\cdot\mathbf{c}| = 2|\cos(\theta_{bc}/2)\cos\theta_{a,b+c}+\sin(\theta_{bc}/2)\cos\theta_{d,b-c}|$$

We may also write the righthand side as twice the absolute value of the two-vector inner product

$$\mathbf{u}_1\cdot\mathbf{u}_2 \equiv (\cos(\theta_{bc}/2),\sin(\theta_{bc}/2))\cdot(\cos\theta_{a,b+c},\cos\theta_{d,b-c})$$

to arrive at the quantum trigonometric identity

$$|\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{d}\cdot\mathbf{b}-\mathbf{d}\cdot\mathbf{c}| = 2(\cos^2\theta_{a,b+c}+\cos^2\theta_{d,b-c})^{1/2}|\cos\theta_{u_1,u_2}|$$

The right sides of these equalities isolate the "classical limiting probability factor" 2 from the second factor, which may achieve its maximum $\sqrt{2}$. Fix any directions **b** and **c**. Then choose **a** relative to $\mathbf{b}+\mathbf{c}$ and choose **d** relative to $\mathbf{b}-\mathbf{c}$ so that $\cos^2\theta_{a,b+c} = 1$ and $\cos^2\theta_{d,b-c} = 1$, respectively. Now we may choose the free directions **b** and **c** to maximize the third factor to $\cos\theta_{u_1,u_2} = \pm 1$. But that means the two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are colinear and hence

$$
\begin{aligned}
\mathbf{u}_1 &= (\cos(\theta_{bc}/2),\sin(\theta_{bc}/2)) = 2^{-1/2}(\cos\theta_{a,b+c},\cos\theta_{d,b-c})\\
&= 2^{-1/2}(\pm 1,\pm 1)
\end{aligned}
$$

and thus the important angle θ_{bc} is seen to be $\pm\pi/2$. More to the point, our identity allows one to exactly trace out the "violation regions" analytically in terms of the trigonometric inner product condition $1 \le |\mathbf{u}_1\cdot\mathbf{u}_2| \le \sqrt{2}$.

Let us summarize the above. One started with a classical probability correlation definition and derived an inequality $|\cdots| \le 2$. The "equality" version of this classical probability version would be in the individual terms

$$|v_i(a)(w_i(b)+w_i(c))+v_i(d)(w_i(b)-w_i(c))| = 2$$

On the other hand, inserting the quantum correlation definition into the left side results in a quantum inequality $|\cdots| \leq 2\sqrt{2}$. Our equality version of this becomes the vector trigonometric identity

$$|\cos\theta_{ab} + \cos\theta_{ac} + \cos\theta_{bd} - \cos\theta_{dc}|$$
$$= 2(\cos^2\theta_{a,b+c} + \cos^2\theta_{d,b-c})^{1/2}|\cos\theta_{u_1,u_2}|.$$

One could call this a quantum spin correlation identity. However, it is also, mathematically, a new result in vector trigonometry in Hilbert space.

4. ZENO (PARMENIDES) THEORY

Because space in this paper is very limited, and because we are giving a mathematically more complete account of our results for the Zeno problem elsewhere (Gustafson, in press), in this and the next section we will restrict ourselves to presenting (1) some little known historical facts about these problems, (2) a quick summary of our investigations to date.

The term Zeno Paradox, i.e., "a watched pot never boils," was introduced in Misra and Sudarshan (1977) to highlight certain fundamental issues in quantum mechanical measurement theory. There is a long history to such quantum measurement problems. For example, Von Neumann (1932) when created his Hilbert space model of quantum mechanics, proved that any given state ϕ of a quantum mechanical system can be steered into any other state ψ in the Hilbert space by an appropriate sequence of very frequent measurements. Thus, in particular, you can freeze a quantum evolution in time by "continually watching it." The allusion to the Greek Zeno refers to a debate which he and Parmenides had with Socrates in Athens approximately 445 BC (Guthrie, 1965). Parmenides, an elderly philosopher, presented to Socrates an interesting proposition to start the debate: reality never changes, therefore motion is not possible. Although Parmenides relied on the younger Zeno for argumentive support, the originating thesis was that of Parmenides. Thus, it might be more accurate to call the quantum versions Parmenides Paradox and Parmenides theory, a suggestion we make here and which we have not seen advanced before.

This author came to these issues through the important and sometimes overlooked paper of Friedman (1972). As you read this paper you find, at least to this author's knowledge, that the first mathematical formulation of the quantum Zeno problem was that of Friedman and his Ph.D. advisor, Ed Nelson at Princeton. Their formulation was motivated to some extent also by an earlier formulation by Feynman (1948) concerned with whether one can determine that a particle trajectory lies within a prescribed space–time region. The analysis of Friedman (1972) leads to the statement there "We may interpret the state of affairs by saying that for almost all times t, if a particle is "continually observed" to determine whether it stays in $\mathcal{E}$, then the probability of an affirmative answer is the same as the probability that the particle was in $\mathcal{E}$ initially." That paper asks if

$$s\text{-}\lim_{n\to\infty}(Ee^{itH_o/n}E)^n$$

exists and is a unitary group. Here E is an orthogonal projection onto a subspace M and H_o a self-adjoint Hamiltonian operator. We (Gustafson, 1974, 1975) worked on this latter question but never published our partial results, except for some mention in Gustafson (1983). Those early results included the following:

- The projected evolution $Z_t = Ee^{itH_o}E$ is a semigroup for all $t \geq 0$ iff M is a proper subspace without regeneration for $U_t = e^{itH_o}$, i.e., $EU_sE^{\perp}U_tE = 0$ for all $t, s \geq 0$. If M is a reducing subspace for U_t, then Z_t is a unitary evolution.
- Assume H_oE has dense domain. Then, EH_oE is a symmetric operator, and EH_oE is self-adjoint if EH_o is a closed operator.

We also had some limited results for the above limit existence question. However, the key issue and requirement was clearly the denseness of $D(H_oE)$.

Recently, we have gone back to our original notes and one can do a little better (Gustafson, 2004):

- $D(H_oE)$ is dense iff EH_o is closeable. Then, $(H_oE)^* = \overline{EH_o} \supset EH_o$ is defined with domain at least as large as $D(H_o)$. Generally $(EH_oE)^* = \overline{EH_o}E$ when EH_o is closeable.

For further recent results, see Gustafson (2004).

In our opinion, one can say that, essentially, all of these mathematical domain-theoretic questions were overlooked or avoided in the early days of the Zeno work. One way to do that is to go to the Heisenberg picture and only work with density matrix (bounded) operators and to state evolutions $\rho(t) = U_t\rho_0U_t^*$. This was the approach of Misra and Sudarshan (1977). However, even in that approach it was necessary to just assume the existence of operator limits such as $s\text{-}\lim_{n\to\infty}\rho_n(t)$ and $s\text{-}\lim_{n\to\infty}T_n(t)$ where $\rho_n(t) = T_n(t)\rho_0T_n^*(t)$ and $T_n(t) = (EU(t/n)E)^n$.

We cannot recount all the Zeno problem literature but do recommend the book Namiki *et al.* (1997), among others.

5. A ZENO ALTERNATIVE

Using this author's (initially, unrelated) results (Gustafson, 2000a) for duals of operator compositions, one may state (Gustafson, 2003b, 2003c)

- Let A be any "continual measurement observable" in the sense that $A = A^*$ is bounded, its range $R(A) \supset D(H_o)$, and $D(H_oA)$ is dense. Then AH_0A is self-adjoint and the exponentiation e^{iAHAt} is unitary.

- More generally, let A, B, C be any densely defined operators in the Hilbert space. Suppose the domains and ranges of the compositions satisfy $D(BC)$ and $D(ABC)$ are dense, $R(BC) \supset D(A)$, $R(C) = D(B)$, $D(BC)^*) \supset R(A^*)$, $D(C^*) \supset R(B^*)$, and C and BC are 1–1. Then, $(ABC)^* = C^*B^*A^*$.

There are other similar technical results that guarantee a reversible, decoherence free, unitary evolution. These results go beyond known Fredholm theory for $(AB)^* = B^*A^*$ which requires closed ranges and finite indices or at least finite defect indices, which to us are not appropriate in most quantum measurement situations.

From these considerations one may propose a Zeno Alternative, in which one considers operators AH_0C, where A and C satisfy the requisite conditions, such as those given above, to assure a continually preconditioned and postconditioned Hamiltonian dynamics that continues to avoid wave function collapse. These A and C, which we call measurors and preparors, are generally more general than von Neumann's measurement theory projections $P = P^* = P^2$ and Ludwig's Effects, e.g., $0 \leq A = A^* \leq 1$. The picture becomes (Gustafson, 2004)

$$\text{Projections} \subset \text{Effects} \subset \text{Measurors/Preparors}.$$

Although the technical sufficiency conditions such as those stated above are rather stringent, such domain and range conditions seem natural to quantum measurement. The physical ansatz is that the measurors and preparors, e.g., A and C, respectively, must have such dense, and compatible to H_o, domains and ranges in order to account for, prepare, respectively, all wave functions ψ in $R(H_o)$, $D(H_o)$, respectively, if one is to be entitled to draw complete conclusions about evolving probabilities $|\psi(t)|^2$.

6. CONCLUSIONS

We have reviewed our work, past and present, on the Bell theory and the Zeno theory from quantum measurement theory.

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